

Periodic cyclic homology of certain nuclear algebras

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Abstract

Relying on properties of the inductive tensor product, we construct cyclic type homology theories for certain nuclear algebras. In this context, we establish continuity theorems. We compute the periodic cyclic homology of the Schwartz algebra of p -adic $GL(n)$ in terms of compactly supported de Rham cohomology of the tempered dual of $GL(n)$.

1. Complete nuclear locally convex algebras

Cyclic type homology groups of an algebra A are computed using chain complexes involving tensor powers of A . When A is a general locally convex algebra, this will involve making a choice of a topological tensor product. A *locally convex algebra* is a locally convex vector space A over \mathbf{C} equipped with a separately continuous multiplication. We shall refer to the *projective* tensor product \otimes_π , the *injective* tensor product \otimes_ϵ , and the *inductive* tensor product \otimes_i . Let E denote a locally convex space. If E is nuclear then $E \otimes_\pi F \simeq E \otimes_\epsilon F$. This is the defining property of nuclear spaces [7, II.34]. The projective tensor product solves the universal problem for continuous bilinear maps; the inductive tensor product solves the universal problem for *separately* continuous bilinear maps. If E is a Fréchet space, then $E \otimes_\pi E \simeq E \otimes_i E$ by [3, III.30, Corollary 1].

Let M be a compact C^∞ -manifold and let $E = C^\infty(M)$ furnished with its standard seminorm topology. Then E is nuclear and Fréchet. Therefore, the class of topologies compatible, in the sense of Grothendieck [7, I.89], with the tensor product structure on $E \otimes E$, is a class with one element. It is with respect to this unique topological tensor product that the cyclic homology of the locally convex unital algebra $C^\infty(M)$ was computed by Connes [5, Ch.II, Theorem 46].

Let $S(G)$ be the Schwartz algebra of a reductive p -adic group. Then $S(G) = \bigcup_K S(G//K)$ in the inductive limit topology, where K is a compact open subgroup of G . The space $S(G)$ is a complete Hausdorff nuclear topological vector space equipped with a separately continuous multiplication. It is the strict inductive limit of unital nuclear Fréchet algebras $S(G//K)$. When we turn to the cyclic homology of $S(G)$, we are faced with a choice of topological tensor product. Topological tensor

products for nuclear spaces such as $S(G)$ are *not* unique [7, II.85]. We choose the completed inductive tensor product $\bar{\otimes}$, as this has good compatibility with strict inductive limits [7, I.76, Prop. 14]. This compatibility is used in a crucial way throughout this Note.

Theorem 1 *Let A be the strict inductive limit of the nuclear Fréchet algebras A_α with $\alpha = 1, 2, 3, \dots$. Then*

- (1) *A is a complete Hausdorff nuclear locally convex algebra*
- (2) *For all $n \geq 1$, $A^{\bar{\otimes} n}$ is a complete Hausdorff nuclear locally convex space and $A^{\bar{\otimes} n} = \varinjlim (A_\alpha^{\bar{\otimes} n})$.*

Proof. (1) We may suppose that the A_α form an increasing sequence of vector subspaces of A such that $A = \bigcup A_\alpha$ as in [7, I.12, I.13]. Now A_α is closed in $A_{\alpha+1}$ by definition [7, I.12] so A_α is closed in A [3, II.32, Prop. 9]. Let $y_n \rightarrow 0$ in A , then (y_n) is a bounded set hence there exists an α for which $y_n \in A_\alpha$ by [3, III.5, Prop. 6]. Let $a \in A$ then $a \in A_\beta$ so take $\gamma = \max(\alpha, \beta)$. Then $a, y_n \in A_\gamma$ and $y_n \rightarrow 0$ in A_γ . So $ay_n \rightarrow 0$ in A_γ by separate continuity of multiplication in A_γ . Then $ay_n \rightarrow 0$ in A , so A is a locally convex algebra. Also A is complete and Hausdorff by [3, II.32, Prop. 9] and nuclear by [7, II.48, Corollaire 1].

(2) Since A_α is Fréchet we have $A_\alpha \bar{\otimes} A_\alpha = A_\alpha \hat{\otimes} A_\alpha$ by [7, I.74], where $\hat{\otimes}$ is the completed projective tensor product. Then $A_\alpha \bar{\otimes} A_\alpha$ is nuclear [7, II.47, Théorème 9] and Fréchet [7, I.43, Prop. 5]. The Corollary in [7, II.70] implies that $(A_\alpha \bar{\otimes} A_\alpha)$ is a strict inductive system. Then $\lim(A_\alpha \bar{\otimes} A_\alpha)$ is complete by [3, II.32, Prop. 9] and so $A \bar{\otimes} A = \lim(A_\alpha \bar{\otimes} A_\alpha)$ by [7, I.76, Prop. 14]. Then $A \bar{\otimes} A$ is the strict inductive limit of nuclear Fréchet spaces hence is a complete Hausdorff nuclear locally convex space, as in (1).

An argument on similar lines shows that $A^{\bar{\otimes} n} = \lim A_\alpha^{\bar{\otimes} n}$ for all $n \geq 1$. Then $A^{\bar{\otimes} n}$ is the strict inductive limit of nuclear Fréchet spaces hence is a complete Hausdorff nuclear locally convex space, as in (1).

2. Cyclic homology

Let A be a locally convex algebra; we do not assume that A has a unit. We denote by $\tilde{A} = \mathbf{C} \oplus A$ the unitization of A . We associate with A the mixed complex $(\bar{\Omega}\tilde{A}, \tilde{b}, \tilde{B})$ of noncommutative differential forms [6], see also [4]. In positive degrees, $\bar{\Omega}^n \tilde{A} = A^{\bar{\otimes} n+1} \oplus A^{\bar{\otimes} n}$. We put $\bar{\Omega}^0 \tilde{A} = A$ and $\bar{\Omega}^n \tilde{A} = 0$ for negative n . The differentials \tilde{b} and \tilde{B} , of degree -1 and $+1$, respectively, are given by

$$\tilde{b} = \begin{pmatrix} b & 1 - \lambda \\ 0 & -b' \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 & 0 \\ N_\lambda & 0 \end{pmatrix}.$$

The continuous differentials b' and b of degree -1 are, for $n > 0$, given by

$$\begin{aligned} b'(a_1 \otimes \cdots \otimes a_n) &= \sum_{i=1}^{n-1} (-1)^{i+1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, \\ b(a_1 \otimes \cdots \otimes a_n) &= b'(a_1 \otimes \cdots \otimes a_n) + (-1)^{n-1} a_n a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

Since the (signed) generator λ of cyclic permutations of $A^{\bar{\otimes} n}$ is continuous, then so is the operator $N_\lambda = \sum_{i=0}^{n-1} \lambda^i$. Thus the differentials \tilde{b} and \tilde{B} are continuous. Moreover, we have that $\tilde{b}^2 = \tilde{b}\tilde{B} + \tilde{B}\tilde{b} = \tilde{B}^2 = 0$.

When defining cyclic type homology theories we shall, *unless the topological tensor product is unique*, indicate explicitly the topological tensor product used. Hochschild homology $HH_*(A, \bar{\otimes})$ of the algebra A , computed with respect to $\bar{\otimes}$, is by definition the homology of the complex $(\bar{\Omega}\tilde{A}, \tilde{b})$. Cyclic homology is defined as $HC_*(A, \bar{\otimes}) = H_*(\text{Tot}\bar{\Omega}\tilde{A}, \tilde{b} + \tilde{B})$, where $\text{Tot}\bar{\Omega}\tilde{A}$ is the total complex of the double complex associated with the mixed complex $(\bar{\Omega}\tilde{A}, \tilde{b}, \tilde{B})$. In degree n we have the finite sum (in the direct sum topology)

$$\text{Tot}_n \bar{\Omega}\tilde{A} = \bigoplus_{p \geq 0} \bar{\Omega}^{n-2p} \tilde{A}.$$

The differential $\tilde{b} + \tilde{B}$ is continuous in this topology. Finally, the *periodic cyclic homology* $HP_*(A, \bar{\otimes})$ of A is the homology of the complex

$$\cdots \xrightarrow{\tilde{b} + \tilde{B}} \bar{\Omega}^{\text{even}} \tilde{A} \xrightarrow{\tilde{b} + \tilde{B}} \bar{\Omega}^{\text{odd}} \tilde{A} \xrightarrow{\tilde{b} + \tilde{B}} \bar{\Omega}^{\text{even}} \tilde{A} \xrightarrow{\tilde{b} + \tilde{B}} \cdots$$

where the spaces of even/odd chains

$$\bar{\Omega}^{\text{even}} \tilde{A} = \prod_{n \geq 0} \bar{\Omega}^{2n} \tilde{A}, \quad \bar{\Omega}^{\text{odd}} \tilde{A} = \prod_{n \geq 0} \bar{\Omega}^{2n+1} \tilde{A}$$

are equipped with the product topology which makes the differential $\tilde{b} + \tilde{B}$ continuous.

Theorem 2 *Let A and A_α be as in Theorem 1. Then*

$$\begin{aligned} HH_*(A, \bar{\otimes}) &= \varinjlim HH_*(A_\alpha) \\ HC_*(A, \bar{\otimes}) &= \varinjlim HC_*(A_\alpha). \end{aligned}$$

Proof. Theorem 1 gives that there is a strict inductive system of mixed complexes $(\bar{\Omega}\tilde{A}_\alpha, \tilde{b}_\alpha, \tilde{B}_\alpha)$ such that, for any $n \geq 0$, $\varinjlim \bar{\Omega}^n \tilde{A}_\alpha = \bar{\Omega}^n(\varinjlim \tilde{A}_\alpha) = \bar{\Omega}^n \tilde{A}$. Using the fact that homology commutes with direct limits [2, p. 28, Prop. 1] and this remark we have

$$\varinjlim HH_*(A_\alpha) = \varinjlim H_*(\bar{\Omega}\tilde{A}_\alpha, \tilde{b}_\alpha) = H_*(\varinjlim \bar{\Omega}\tilde{A}_\alpha, \varinjlim \tilde{b}_\alpha) = H_*(\bar{\Omega}\tilde{A}, \tilde{b}) = HH_*(A, \bar{\otimes})$$

where $\tilde{b} = \varinjlim \tilde{b}_\alpha$. This differential is continuous by Theorem 1. Continuity of cyclic homology is proved in the same way when we use the fact that direct limits commute with direct sums.

Theorem 3 *Let A and A_α be as in Theorem 1. Assume that there exists $N > 0$ such that $HH_n(A_\alpha) = 0$ for all $n > N$ and all α . Then*

$$HP_*(A, \bar{\otimes}) = \varinjlim HP_*(A_\alpha).$$

Proof. Let D be a locally convex algebra such that $HH_n(D) = 0$ for all $n > N$. Then $HP_{\text{even}}(D) = HC_{2n}(D)$, $HP_{\text{odd}}(D) = HC_{2n+1}(D)$ for any n such that $2n$ and $2n+1$ are greater than N . Indeed, let us define a map $T : HP_{\text{even}}(D) \rightarrow HC_{2n}(D)$ by

$$T : [f] \mapsto [(f_0, \dots, f_{2n})], \quad f_{2i} \in \bar{\Omega}^{2i} \tilde{D},$$

for any cycle $f = \{f_{2n}\}_{n \geq 0}$ in $\bar{\Omega}^{\text{even}} \tilde{D}$. It is clear that T maps even periodic cycles to cycles in $\text{Tot}_{2n} \bar{\Omega} \tilde{D}$.

T is surjective, for let us take a cycle $f = (f_0, \dots, f_{2n})$ in $\text{Tot}_{2n} \bar{\Omega} \tilde{D}$. Having embedded f in $\bar{\Omega}^{\text{even}} \tilde{D}$, we calculate that $\tilde{b}\tilde{B}f_{2n} = -\tilde{B}\tilde{b}f_{2n} = \tilde{B}^2 f_{2n-2} = 0$ so that $g_{2n+1} = \tilde{B}f_{2n} \in \bar{\Omega}^{2n+1} \tilde{D}$ is a cycle in the Hochschild complex. Since Hochschild homology vanishes for $2n+1 > N$, there exists $f_{2n+2} \in \bar{\Omega}^{2n+2} \tilde{D}$ such that $\tilde{b}f_{2n+2} = -g_{2n+1}$. Then $\tilde{b}f_{2n+2} + \tilde{B}f_{2n} = 0$. Proceeding this way we construct a cycle F in $\bar{\Omega}^{\text{even}} \tilde{D}$ such that $T(F) = f$.

The map T is also injective. Let $F = \{f_{2n}\}_{n \geq 0}$ be a cycle in $\bar{\Omega}^{\text{even}} \tilde{D}$. Then $[T(F)] = 0$ in $HC_{2n}(D)$ if and only if there exists a chain $h = (h_1, \dots, h_{2n+1}) \in \text{Tot}_{2n+1} \bar{\Omega} \tilde{D}$ such that $(\tilde{b} + \tilde{B})h = T(F)$. Then

$$\tilde{b}(f_{2n+2} - \tilde{B}h_{2n+1}) = \tilde{b}f_{2n+2} + \tilde{B}\tilde{b}h_{2n+1} = -\tilde{B}f_{2n} + \tilde{B}(f_{2n} + \tilde{B}h_{2n-1}) = 0$$

and so there exists $h_{2n+3} \in \bar{\Omega}^{2n+3} \tilde{D}$ such that $\tilde{b}h_{2n+3} + \tilde{B}h_{2n+1} = f_{2n+2}$. This procedure yields a chain $H \in \bar{\Omega}^{\text{odd}} \tilde{D}$ such that $(\tilde{b} + \tilde{B})H = F$, and so $[F] = 0$ in $HP_{\text{even}}(D)$, proving that T is injective. Note that $HC_{2n}(D) \simeq HC_{2m}(D)$ for all n, m such that $2n, 2m > N$. The argument is the same in the odd case.

Returning to the proof of the theorem, we first use Theorem 2 to show that $HH_n(A, \bar{\otimes}) = 0$ for all $n > N$. Then using the above remark and continuity of HC we write

$$\varinjlim HP_{\text{even}}(A_\alpha) = \varinjlim HC_{2n}(A_\alpha) = HC_{2n}(A, \bar{\otimes}) = HP_{\text{even}}(A, \bar{\otimes}),$$

provided $2n > N$. The proof of the odd case is the same.

3. Periodic cyclic homology of $S(GL(n))$

Let F be a non-archimedean local field and let $G = GL(n) = GL(n, F)$. Let K be a compact open subgroup of G . Define $S(G//K)$ to be all functions $f : G \rightarrow \mathbf{C}$ which are K -bi-invariant and rapidly decreasing. Then $S(G//K)$, in its standard seminorm topology, is a unital nuclear Fréchet algebra. The Schwartz algebra $S(G)$ is given by $S(G) = \bigcup_K S(G//K)$ in the inductive limit topology [13]. The algebras $S(G)$, $S(G//K)$ satisfy the conditions of Theorem 1. By Mischenko's theorem [9], the Fourier transform determines an isomorphism of unital Fréchet algebras:

$$S(G//K) \simeq \bigoplus_M [C^\infty(F(M : K))]^{W(M:K)}$$

where $F(M : K) \rightarrow E_2(M : K)$ is the complex Hermitian vector bundle of K -fixed vectors in the induced Hilbert bundle $F(M) \rightarrow E_2(M)$. The vector bundle $F(M : K)$ is trivialized. One Levi subgroup M is chosen in each G -conjugacy class.

We are led to the following issue. Let X be a compact smooth manifold (in fact a compact torus), W a finite group acting on X , and $F \rightarrow X$ a (trivialized) complex Hermitian vector bundle such that F is a W -bundle. The group W acts via intertwining operators $a(w : x) : F_x \rightarrow F_{wx}$. For the group $GL(n)$, normalized intertwining operators [12] may be chosen such that each isotropy subgroup W_x acts trivially in the fibre F_x , i.e. $w \in W_x$ implies $a(w : x) = 1$.

We consider $B = C^\infty(X)^W$, $A = [C^\infty(\text{End } F)]^W$, $E = C^\infty(F)^W$. It is elementary to check that E is an $A - B$ -bimodule. Then we have a map $\Phi : A \rightarrow \text{End}_B(E)$.

Lemma 4 *Let $v \in F_x$. Then there exists an invariant section s of F such that $s(x) = v$.*

Proof. Choose a smooth section t such that $t(x) = |W_x|^{-1}v$ and $\text{supp } t$ does not contain any point in the orbit Wx except x . Now average t by defining $s = \sum_{w \in W} wt$. Then s is an invariant smooth section such that

$$s(x) = \sum_{w \in W} a(w : x)t(w^{-1}x) = \sum_{w \in W_x} a(w : x)t(x) + \sum_{w \notin W_x} a(w : x)t(w^{-1}x) = |W_x|t(x) = v.$$

Lemma 5 *The map $\Phi : A \rightarrow \text{End}_B(E)$ is an isomorphism of Fréchet algebras.*

Proof. Injectivity of the map Φ follows from Lemma 4. To prove surjectivity, define, for $H \in \text{End}_A(E)$, $f_H(x)(v) = (Hs_v)(x)$ with $v \in F_x$, s_v an invariant section through v . Given that $(Hs_v)(x) = H(x)s_v(x)$, this definition is independent of the choice of the section s_v . It is elementary to check that $f_H \in A$.

By Lemma 5, the algebras A and B are Morita equivalent Fréchet algebras, which implies that they have the same Hochschild homology [8, p. 194]. We now have that

$$HH_*(S(G//K)) = \bigoplus_M HH_*(C^\infty(X(M : K)))^{W(M:K)}$$

Implicit in the proof of Lemma 45, p. 344 of [5] is the identification of Hochschild homology of the algebra $C^\infty(V)$ with the differential forms on V . Noting the perfect duality between the complexes of forms and currents [11, p. 44-45] and using the invariance result in [14, p. 240], the Hochschild homology of $W(M : K)$ -invariant smooth functions on the smooth manifold $X(M : K)$ may be identified with the $W(M : K)$ -invariant differential forms on $X(M : K)$. Given that $\dim X(M : K) \leq n$ for all compact open subgroups K of $GL(n)$, we have that $HH_p(S(G//K)) = 0$ for all $p > n$ and all such K . Using Theorem 3 we have established the following result.

Theorem 6

$$HP_*(S(G), \bar{\otimes}) = \varinjlim HP_*(S(G//K))$$

We remark that the homology theory on the right is the same as the theory $h_*(S(G))$ of [1].

Each quotient space $X(M : K)/W(M : K)$ creates a disjoint union of compact orbifolds, which together form the tempered dual of $GL(n)$. Each orbifold is the quotient of a compact torus \mathbf{T}^k by a product of symmetric groups and we have $k \leq n$ [10]. By the de Rham cohomology of an orbifold X/W we shall mean the W -invariant part of the de Rham cohomology of X . We first apply the invariance result in [14, p. 240] and then apply the fundamental result of Connes [5, Ch. II, Theorem 46] to obtain the following theorem.

Theorem 7 *The periodic cyclic homology $HP_0(-, \bar{\otimes})$ (resp. $HP_1(-, \bar{\otimes})$) of the Schwartz algebra $S(GL(n))$ is isomorphic to the compactly supported even (resp. odd) de Rham cohomology of the tempered dual of $GL(n)$.*

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